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CONVEX SURFACES WHICH INTERSECT EACH CONGRUENT COPY OF THEMSELVES IN A CONNECTED SET

BY
PETER MANI-LEVITSKA
Mathematische Institut, Universität Bern, 3012 Bern, Switzerland

ABSTRACT
Let $K \subset \mathbf{R}^3$ be a three-dimensional convex body such that, for every isometry ρ of $\mathbf{R}^3$, the boundaries of K and ρK meet in a connected set. Then K is a parallel set of some possibly degenerate linesegment.

1. Introduction

Unless otherwise stated, we shall adopt the terminology of B. Grünbaum's book [6]. By a convex body in $\mathbf{R}^3$ we understand a compact convex subset of $\mathbf{R}^3$ with nonempty interior.

DEFINITION 1. A convex body K in $\mathbf{R}^3$ has the intersection property, if for every isometry ρ of $\mathbf{R}^3$, the boundaries $\mathrm{bd}(K)$ and $\mathrm{bd}(\rho K)$ have a connected intersection.

It was asked by T. Bonnesen and W. Fenchel whether the Euclidean balls are characterized among all convex bodies in $\mathbf{R}^3$ by the intersection property, see page 141 of [3]. However, as H. Hadwiger [7] pointed out, each parallel body of a linesegment enjoys the same property. The question, whether there are any other examples, has been posed again by J. J. Schäffer [10], and shall be answered in the negative here. There are a number of further problems related to this result, two of which have been solved recently. P. Goodey [5] shows that a convex body K in $\mathbf{R}^n$ for which $\mathrm{bd}(K) \cap \mathrm{int}(gK)$ is a topological ball whenever g is a rigid motion, must be a Euclidean ball. Ch. Senn [11] characterizes the vector sums of a ray and a Euclidean ball as the only unbounded closed convex sets in $\mathbf{R}^3$ with the intersection property.

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2. Strictly convex bodies

Our main purpose here is a proof of the following result. Remember that a convex body K is called strictly convex, if every boundary point of K is an exposed point of K.

PROPOSITION 1. *Let K be a convex body in $\mathbf{R}^3$. If K is strictly convex and has the intersection property, then K is a Euclidean ball.*

Let us first introduce some additional notation. If v is a unit vector in $\mathbf{R}^3$ we set $v^\perp = \{x \in \mathbf{R}^3 : \langle x, v\rangle = 0\}$ and denote by $\pi_v : \mathbf{R}^3 \to v^\perp$ the orthogonal projection onto $v^\perp$. The shadow boundary of a convex body K above the plane E is the set $\Sigma(K,E)$ of all points $p \in K$ for which $\pi_v(p) \in \operatorname{relbd}(\pi_v[K])$, where v is a unit vector with $v^\perp = E$.

DEFINITION 2. Let K and L be closed convex sets in $\mathbf{R}^3$, having the same affine hull E. We say that K surrounds L at the point $p \in E$, if there is an open neighbourhood U of p in E such that $L \cap U$ is a proper subset of $K \cap U$, and $(\operatorname{relbd} K) \cap L \cap U$ is compact and contains the point p.

With this notion we can express our basic condition for a convex body to have the intersection property.

LEMMA 1. *Let K be a convex body in $\mathbf{R}^3$. Assume that there is an isometry ρ of $\mathbf{R}^3$ and a boundary point p of K such that ρK surrounds K at p. Then K does not have the intersection property.*

PROOF. Let U be an open neighbourhood of p in $\mathbf{R}^3$ such that $(\rho K \cap U) \supset (K \cap U)$, and $(\operatorname{bd}\rho K) \cap K \cap U$ is compact, with $p \in (\operatorname{bd}\rho K) \cap K \cap U \neq \operatorname{bd}\rho K$. There is a point q in $(\operatorname{bd}\rho K \cap U)\setminus K$, and therefore we have $L^3(\rho K \cap U) > L^3(K \cap U)$, where L^3 denotes the Lebesgue measure in $\mathbf{R}^3$. Since ρK has the same Lebesgue measure as K, we cannot have $\rho K \supset K$, thus we find a point r in $(\operatorname{bd}\rho K \cap \operatorname{bd} K)\setminus U$. r and p lie in different components of $\operatorname{bd}\rho K \cap \operatorname{bd} K$, and K does not have the intersection property.

PROOF OF PROPOSITION 1. Let us assume that K is a counterexample to Proposition 1. An elementary argument shows that no ellipsoid other than a Euclidean ball has the intersection property, so at least one shadow boundary $\Sigma(K,E)$ of K is not contained in a plane [2]. We may assume $E = \mathbf{R}^2 = \operatorname{lin}\{e_1, e_2\}$ where we set $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Denoting by s the reflection at the plane $\mathbf{R}^2$ we set, for $\tau \in \mathbf{R}$, $K(\tau) := s[K] + \tau e_3$. For $p \in D := \pi_{e_3}[K]$ we set

$$h(p) := \inf\{\lambda : p + \lambda e_3 \in K\} \text{ and } h(\tau)(p) := \inf\{\lambda : p + \lambda e_3 \in K(\tau)\}, \quad \tau \in \mathbf{R}.$$

We consider the sets $L(\tau) := \{p \in D : h(p) < h(\tau)(p)\}$ and $M(\tau) := \{p \in D : h(p) > h(\tau)(p)\}$, $\tau \in \mathbf{R}$. Notice that, for $\tau \leqq \sigma$, we always have $L(\tau) \subset L(\sigma)$, $M(\tau) \supset M(\sigma)$. Set

$$\tau_1 := \inf\{\langle x, e_3\rangle : x \in \Sigma(K, \mathbf{R}^2)\} \quad \text{and} \quad \tau_2 := \sup\{\langle x, e_3\rangle : x \in \Sigma(K, \mathbf{R}^2)\}.$$

Then, clearly, $L(\tau) \cap \operatorname{relbd}(D) \neq \emptyset$, $M(\tau) \cap \operatorname{relbd}(D) \neq \emptyset$, whenever $2\tau_1 < \tau < 2\tau_2$. Consider a unit vector $u \in \mathbf{R}^2$, and let us say that $X \subset \mathbf{R}^2$ is dominated by D in direction u, if $X \subset \operatorname{int} D + R$, where $R = \{\lambda u : \lambda \geqq 0\}$ is the ray issuing from the origin in direction u. We want to show

(1) there are a number τ with $2\tau_1 < \tau < 2\tau_2$, a component C of $L(\tau) \cup M(\tau)$, and a unit vector $u \in \mathbf{R}^2$ such that $\operatorname{cl}(C)$ is dominated by D in direction u.

Let us say that a subset N of the circle $S^1 = \{x \in R^2 : \|x\| = 1\}$ is spanning, if $0 \in \operatorname{conv}(N)$ but $0 \notin \operatorname{conv}(M)$ for any $M \subset N$, $M \neq N$. A spanning set $N \subset S^1$ then consists of 2 or 3 elements. A vector $n \in S^1$ is called an outer normal of D at $p \in \operatorname{relbd} D$, if $\langle n, p\rangle \geqq \langle n, q\rangle$ for all $q \in D$. We now choose $\tau \in (2\tau_1, 2\tau_2)$ and a component C of $L(\tau)$. If τ and C do not satisfy (1), we find a spanning subset N of S^1 and, for every $n \in N$, a point $p(n) \in \operatorname{cl}(C) \cap \operatorname{relbd} D$ such that n is an outer normal of D at $p(n)$. Since $\operatorname{cl}[L(\tau) \cap \operatorname{relbd} D] \subset L(\sigma)$ for all σ with $\tau < \sigma < 2\tau_2$, we may assume, by enlarging τ if needed, that $\{p(n) : n \in N\}$ is contained in $C \cap \operatorname{relbd}(D)$.

Set $Y := \{\lambda n : n \in N, \lambda \in [0,1]\}$. C is an open connected subset of D, so there exists an injective continuous map $\phi : Y \to C$ with $\phi(n) = p(n)$ for all $n \in N$ and $\operatorname{Im}(\phi) \cap \operatorname{relbd}(D) = \{p(n) : n \in N\}$. Since $\tau \in (2\tau_1, 2\tau_2)$, we have $M(\tau) \neq \emptyset$. Choose a component A of $M(\tau)$, and notice $A \cap \operatorname{Im}(\phi) = \emptyset$. By the definition of ϕ and because $D = \pi_{e_3}[K]$ inherits the strict convexity from K, we find $u \in S^1$ such that $\operatorname{cl}(A)$ is dominated by D in direction u, and (1) follows.

Let us set, for $\tau \in \mathbf{R}$ and $x \in \mathbf{R}^2$,

$$K(\tau, x) := K(\tau) + x, D(\tau, x) := \pi_{e_3}[K(\tau, x)],$$

and denote by $h(\tau, x) : D(\tau, x) \to \mathbf{R}$ the height function, given by

$$h(\tau, x)(p) := \inf\{\lambda : p + \lambda e_3 \in K(\tau, x)\}, \qquad p \in D(\tau, x).$$

By an arc we understand as usual a homeomorphic image of some compact interval $I \subset \mathbf{R}$.

Now we choose τ, C and u according to (1), and assume first that C is a component of $L(\tau)$, with $C \cap \operatorname{relbd}(D) \neq \emptyset$.

$$C_0 := \{p \in C : h(\tau)(p) - h(p) \geqq h(\tau)(q) - h(q) \text{ for all } q \in C\}$$

is a compact subset of C, so we find a pair $\{s,t\}$ of points in $C \cap \operatorname{relbd}(D)$ and arcs $P \subset C \setminus C_0$, $Q \subset \operatorname{cl}(\mathbf{R}^2 \setminus D)$, both with end points $\{s,t\}$, such that $\{s,t\} = P \cap \operatorname{relbd}(D) = Q \cap D$, Q is dominated by D in direction u, and C_0 belongs to the bounded component of $\mathbf{R}^2 \setminus (P \cup Q)$.

By the definition of C_0 and P, there exists a number $\sigma \in (-\infty, \tau)$ such that $h(\sigma)(q) > h(q)$ for all $q \in C_0$, but $h(\sigma)(p) < h(p)$ for all $p \in P$. Further, if we choose $\beta > 0$ sufficiently small, we have $P \subset D(\sigma, \beta\mu)$ and $h(\sigma, \beta u)(p) < h(p)$ for all $p \in P$, whereas $h(\sigma, \beta u)(q) > h(q)$, $q \in C_0$. Let us also ensure that the arc Q mentioned above satisfies $(Q \setminus \{s,t\}) \subset (D(\sigma, \beta u) \setminus D)$. $J := P \cup Q$ is a closed Jordan curve contained in $D(\sigma, \beta u)$. For each point $p \in J$ we have either $K \cap \pi_{e_3}^{-1}[p] = \emptyset$ or $h(p) > h(\sigma, \beta u)(p)$, whereas there is some q in the bounded component B of $\mathbf{R}^2 \setminus J$ such that $q \in D$ and $h(q) < h(\sigma, \beta u)(q)$. Choose $q_0 \in B \cap D$ such that

$$\delta := h(\sigma, \beta u)(q_0) - h(q_0) \geqq h(\sigma, \beta u)(q) - h(q) \qquad \text{for all } q \in B \cap D.$$

Then $K(\sigma, \beta u)$ surrounds $K + \delta e_3$ at $q_0 + (h(q_0) + \delta)e_3$, contradicting Lemma 1. The case $C \subset L(\tau)$, $C \cap \operatorname{relbd}(D) = \emptyset$, is trivial. We find a Jordan curve $J \subset C$ and a number $\sigma \in \mathbf{R}$ with $h(p) > h(\sigma)(p)$ for all $p \in J$ and $h(q) < h(\sigma)(q)$ for some q in the bounded component of $\mathbf{R}^2 \setminus J$. Finally, if C is a component of $M(\tau)$, we just interchange the roles of K and $s[K]$. Thus Proposition 1 is established.

3. Bodies which are not strictly convex

Throughout this section we assume that $K \subset \mathbf{R}^3$ is a convex body, containing a linesegment S in its boundary. If $u \in \mathbf{R}^3$ is a unit vector, we denote by $I(u)$ the set of all affine isometries $\rho : \mathbf{R}^3 \to \mathbf{R}^3$ which preserve the scalar product with u, such that $\langle \rho x, u \rangle = \langle x, u \rangle$ for all $\rho \in I(u)$ and $x \in \mathbf{R}^3$. Furthermore we set, for $\tau \in \mathbf{R}$, $K(u, \tau) := \{p \in K : \langle p, u \rangle = \tau\}$. Recall that u is an outer normal of K at $p \in \operatorname{bd} K$, if $\langle u, p \rangle \geqq \langle u, q \rangle$ for all $q \in K$.

LEMMA 2. *With the above notation, let u be an outer normal of K at some point $p \in \operatorname{relint} S$. Assume that, given any neighbourhood N of 0 in $\mathbf{R}$, there are a number $\tau < 0$ in N and an isometry $\rho \in I(u)$ such that, with $\tau_0 := \langle p, u \rangle + \tau$,*

(2) $\operatorname{relint} S \cap \operatorname{relint} \rho S$ *consists of exactly one point,*

(3) $[\operatorname{relint} K(u,\tau_0)]\backslash \rho K(u,\tau_0)$ *and*
$[\operatorname{relint} \rho K(u,\tau_0)]\backslash K(u,\tau_0)$ *are both connected.*

Then K does not have the intersection property.

PROOF. Call $q \in \operatorname{bd}(K)$ a smooth point of K if there is a unique outer normal of K at q, and call q a vertex of K if the set of all outer normals of K at q is twodimensional. Notice first

(4) if K has the intersection property, then there exists no vertex of K.

Otherwise let q_1 be a vertex of K. Let $A_2 \subset K$ be a maximal Euclidean ball contained in K, and choose $q_2 \in A_2 \cap \operatorname{bd}(K)$. By definition of q_1 there is a Euclidean ball A_1 which surrounds K at q_1, and which has a smaller radius than A_2. Hence we find an isometry ρ of $\mathbf{R}^3$ such that $\rho q_1 = q_2$ and $\rho A_1 \subset A_2$. K surrounds ρK at q_2, which is a contradiction to Lemma 1, and (4) follows. Similarly

(5) if K has the intersection property, then there exists no twodimensional face of K.

Now let $p \in \operatorname{relint}(S)$ and the unit vector $u \in \mathbf{R}^3$ satisfy the assumptions of Lemma 2. We can suppose that p is the origin 0 of $\mathbf{R}^3$.

(6) We may also assume $0 = \pi_u(p) \in \operatorname{relint} \pi_u(K)$.

If 0 is a smooth point of K, (6) is satisfied. So let us assume that H_1 and $H_2 \neq H_1$ are supporting planes of K, with $0 \in H_1 \cap H_2$, and denote by H_i^+ the closed halfspace bounded by H_i and containing K. There exists some outer normal v of K at 0 such that $0 \in \operatorname{relint}(\pi_v(K))$. We also choose a unit vector w such that $\operatorname{lin}\{w\}$ and $\operatorname{aff} S$ are parallel lines. If we find, in every neighbourhood N of 0 in $\mathbf{R}$, some number $\tau < 0$ such that $K(v,\tau)$ does not contain a linesegment T whose affine hull is parallel to $\operatorname{aff} S$, then v satisfies (2) and (3). The corresponding isometry $\rho \in I(v)$ consists of a translation $x \mapsto x + \lambda w$, $\lambda \neq 0$, small, followed by a rotation with axis $\operatorname{lin}\{v\}$ and a sufficiently small angle. Otherwise, since $\pi_w(K)$ has at most countably many vertices, we find $\tau < 0$ in N such that $K(v,\tau)$ contains a linesegment T with $\operatorname{aff}(T)$ parallel to $\operatorname{aff} S$, and such that every $q \in \operatorname{relint}(T)$ is a smooth point of K. Choose $q \in \operatorname{relint}(T)$ arbitrarily, denote by H the unique supporting plane of K containing q, and by H^+ the closed halfspace which contains K and is bounded by H. Let r be an arbitrary endpoint of the linesegment S.

Let γ be an isometry of $\mathbf{R}^3$, such that $\gamma(r)=q$, $\gamma(H_1^+\cap H_2^+)\subset H^+$, $H\cap\gamma(H_1^+\cap H_2^+)=\mathrm{aff}(T)$. Denoting by $L\subset H$ the line which is orthogonal to $\mathrm{aff}(T)$ and contains q, we choose a rotation β with axis L such that $\mathrm{relint}(\beta\gamma S)\subset \mathrm{int}(H^+)$. Let α be a translation with $\alpha(H)\subset\mathrm{int}(H^+)$. If the angle of β is small enough, and α is chosen properly, the convex body K surrounds $\alpha\beta\gamma(K)$ at some point q' close to q. So, according to Lemma 1, K does not have the intersection property, and it remains to consider the case where (6) holds.

(7) We may assume $S=\pi_u(S)\subset\mathrm{relint}(\pi_u K)$.

In view of (6) we find, if (7) does not hold, an endpoint r of S and a supporting hyperplane H of K with $r\in H$ and $S\not\subset H$. If $0\in\mathrm{relint}(S)$ is not a smooth point of K, r is a vertex of K, a case which may be excluded by (4). If 0 is smooth we denote by L the supporting plane of K which contains 0 and denote by H^+ and L^+ the closed halfspaces bounded by H and L containing K. By (5) we may assume $H\cap L\cap K=\{r\}$. As above we find an isometry γ of $\mathbf{R}^3$ such that

$$\gamma(r)=0,\quad \gamma(H^+\cap L^+)\subset L^+,\quad L\cap\gamma(H^+\cap L^+)=\mathrm{aff}(S).$$

Then K surrounds γK at q, and this case can again be excluded by Lemma 1. In view of (7) we find a neighbourhood N of 0 in $\mathbf{R}$ such that

(8) $\pi_u(K(u,\sigma))\subset\mathrm{relint}\,\pi_u(K)$, for every $\sigma\in N$.

Let us choose $\tau<0$ in N and $\rho\in I(u)$ such that they satisfy the conditions (2) and (3) of our lemma. If r_1 and r_2 are the endpoints of the linesegment S, we denote by $C(r_i)$ the unique bounded component of $\mathbf{R}^3\backslash(\mathrm{bd}\,K\cup\mathrm{bd}\,\rho K)$ which contains r_i in its closure. Then we may assume that

(9) one of the sets $C(r_i)$, $i\in\{1,2\}$, satisfies

$$\langle x,u\rangle\in(0,\tau),\qquad \text{all } x\in C(r_i).$$

Otherwise we find piecewise linear maps $\phi_i:[0,1]\to K$ such that $\phi_i(0)=r_i$, $\langle\phi_i(1),u\rangle=\tau$, $\mathrm{Im}\,\phi_i\subset\{r_i\}\cup C(r_i)$, and such that there exists, for every $\xi\in(0,1]$, a number $\sigma\in[\tau,0)$ satisfying $\phi_i(\xi)\in[\mathrm{relint}\,K(u,\sigma)]\backslash[\rho K(u,\sigma)]$. By condition (3) there is also a piecewise linear map $\psi:[0,1]\to[\mathrm{relint}\,K(u,\tau)]\backslash[\rho K(u,\tau)]$ with $\psi(0)=\phi_1(1)$ and $\psi(1)=\phi_2(1)$. We may assume that $A=\pi_u[\mathrm{Im}(\phi_1)\cup\mathrm{Im}(\phi_2)\cup\mathrm{Im}(\psi)]$ is disjoint to $S\cap\rho S$, hence there exists a Jordan curve $J\subset A\cup S$ containing the point of $S\cap\rho S$. Setting, for $x\in\mathrm{Im}(\pi_u)$ and a convex body $B\subset\mathbf{R}^3$,

$$h(B)[x]:=\inf\{\lambda:x-\lambda u\in B\}\ \text{if } x\in\pi_u(B),\ \text{and } h(B)[x]:=\infty\ \text{otherwise}$$

we derive from (8) that $K(u,\xi)\subset K(u,\eta)$, whenever $0\geqq\xi\geqq\eta\geqq\tau$, and therefore, by the construction of J,

(10) $h(K)[x]\leqq h(\rho K)[x]$, for every $x\in J$.

Let D be the bounded component of $\mathrm{Im}(\pi_u)\setminus J$. By (2) there is a point $q\in D\cap\rho S$, and obviously $0=h(\rho K)[q]<h(K)[q]$. Setting $\mu:=\sup\{h(K)[q]-h(\rho K)[q]:q\in \mathrm{cl}D\}$ we have $\mu>0$ and derive from (10) that $h(K)[q_0]-h(\rho K)[q_0]=\mu$, for some $q_0\in D$. Again using (10) we see that K surrounds $\rho K-\mu u$ at the point $q_0-(h(K)[q_0])u$. By Lemma 1, K would not have the intersection property. So (9) is established. Assume, then, $\langle x,u\rangle\in(0,\tau)$ for every $x\in C(r_1)$. Consider a point x in the boundary of $\pi_u(C(r_1))\subset\mathrm{Im}(\pi_u)$, and set $y=x-(h(K)[x])u$. By (8) we have $\langle u,w\rangle>0$, for every outer normal vector w of K at y. Hence $y\in\mathrm{bd}(\rho K)$, or else x would belong to $\pi_u(C(r_1))$ itself, rather than to its boundary. We conclude that $h(K)[x]=h(\rho K)[x]$, whereas, obviously, $0=h(K)[r_1]<h(\rho K)[r_1]$. Since $r_1\in\pi_u(C(r_1))$ we find, as above, a point $q_0\in\pi_u(C(r_1))$ and a number $\mu>0$ such that ρK surrounds $K-\mu u$ at $q_0-(h(\rho K)[q_0])u$. By Lemma 1, K does not have the intersection property, and Lemma 2 is established.

Recall that $K\subset\mathbf{R}^3$ is a convex body having a linesegment S in its boundary.

Proposition 2. *If K has the intersection property, there exist a unit vector $u\in\mathbf{R}^3$ and a number $\alpha>0$ such that*

(11) *every point p in the shadow boundary $\Sigma(K,\mathrm{Im}(\pi_u))$ lies in a linesegment $L\subset\mathrm{bd}\,K$ of length α, whose affine hull is a parallel to* $\mathrm{lin}\{u\}$,

(12) *whenever $L\subset\mathrm{bd}\,K$ is a linesegment, $\mathrm{aff}\,L$ is a parallel to* $\mathrm{lin}\{u\}$.

Proof. Choose the unit vector $u\in\mathbf{R}^3$ such that $\mathrm{lin}\{u\}$ is parallel to $\mathrm{aff}\,S$, where S is some linesegment in $\mathrm{bd}\,K$. If a and b are points in $\mathbf{R}^3$ we set, for $T=\mathrm{conv}\{a,b\}$, $\lambda(T):=\|a-b\|$. Define $\alpha>0$ by

$$\alpha:=\sup\{\lambda(K\cap\pi_u^{-1}[p]):p\in\mathrm{relbd}\,\pi_u(K)\}.$$

The set $X:=\{p\in\mathrm{relbd}\,\pi_u(K):\lambda(K\cap\pi_u^{-1}[p])=\alpha\}$ is nonvoid and compact. We want to show

(13) X is open in $\mathrm{relbd}\,\pi_u(K)$.

Otherwise there is $p\in X$ and a sequence $(q_i)_{i\in\mathbf{N}}$ of points $q_i\in\mathrm{relbd}\,\pi_u(K)$, converging to p, such that

$$\lambda(K \cap \pi_u^{-1}[q_i]) = : \beta_i < \alpha, \qquad \text{for every } i.$$

We may assume that $u = (1,0,0)$, p is the origin 0 of $\mathbf{R}^3$ and belongs to relint S, $S = K \cap \pi_u^{-1}[p]$, and $v = (0,0,-1)$ is an outer normal of K at 0. Let N be a neighbourhood of 0 in $\mathbf{R}$. We find a number $\tau < 0$ in N and an index i such that $\langle q_i, v\rangle = \tau$. Choose γ in the open interval (β_i, α) and define the translation ϕ by $\phi(x) = x + \gamma u$. Since $\lambda(K \cap \pi_u^{-1}[q_i]) = \beta_i < \gamma$ and $\langle q_i, v\rangle = \tau$, one of the components of $\operatorname{relbd}(K(v,\tau)) \cap \operatorname{relbd}(\phi K(v,\tau))$ consists of a single point, whereas $S \cap \phi S$ is a nondegenerate linesegment. We can choose a point $m \in \operatorname{relint}(S \cap \phi S)$ and a rotation ρ with axis $m + \operatorname{lin}\{v\}$ and sufficiently small angle $\varepsilon \neq 0$ such that $[\operatorname{relint}(K(v,\tau))] \backslash \rho\phi(K(v,\tau))$ and $[\operatorname{relint}(\rho\phi(K(v,\tau)))] \backslash K(v,\tau)$ are both connected. Here, if one component of $\operatorname{relbd}(K(v,\tau)) \cap \operatorname{relbd}(\phi K(v,\tau))$ is a linesegment, we have to choose the sign of ε appropriately. Then $v, S, 0, N, \tau$ and $\rho\phi$ fulfil the requirements of Lemma 2, and K would not have the intersection property. So (13) follows. Since X is also compact and nonvoid it coincides with $\operatorname{relbd}(\pi_u(K))$, and (11) is established. If there were a linesegment $L \subset \operatorname{bd} K$ with $\operatorname{aff} L$ parallel to $\operatorname{lin}\{v\}$ for some unit vector $v \notin \{u, -u\}$, we could apply the analogue of the above argument with v replacing u and would conclude that K has a twodimensional face. But this contradicts Lemma 1, compare (5) in the proof of Lemma 2. The relation (12), and Proposition 2, are thus established.

Recall that $B \subset \mathbf{R}^3$ is a Euclidean ball if

$$B = \{y \in \mathbf{R}^3 : \| y - p \| \leqq \rho\}, \qquad \text{for some } p \in \mathbf{R}^3 \quad \text{and} \quad \rho > 0,$$

and that a Euclidean disc is the intersection of some Euclidean ball B with a plane containing its midpoint. $Y \in \mathbf{R}^3$ is called a parallel body of the compact convex set X, if there is a Euclidean ball B, centered at the origin, such that $Y = X + B$. For the proof of the next result we shall use a few elementary facts concerning the curvature of planar convex sets. The reader may consult [1,2,4] for a general and thorough exposition of these matters.

Proposition 3. *Let $K \subset \mathbf{R}^3$ be a convex body which has the intersection property. If* bd K *contains a linesegment, then K is a parallel body of some linesegment.*

Proof. Choose a unit vector $u \in \mathbf{R}^3$ and the number $\alpha > 0$ such that K, u and α satisfy Proposition 2. Then

(14) $\pi_u K$ is a Euclidean disc.

Otherwise let D be its circumscribed disc, the smallest Euclidean disc containing $\pi_u K$. Then there is an arc $R \subset \operatorname{relbd}(\pi_u K)$ whose endpoints r_1, r_2 belong to relbd(D) and whose inner points all lie in relint D. By translating D in a direction orthogonal to aff$\{r_1, r_2\}$ we obtain a disc D_1, congruent to D, which is surrounded by $\pi_u K$ at some point $p_1 \in R$. Similarly there is a disc D_2, congruent to D, which surrounds $\pi_u K$ at some point p_2. This allows us to find an isometry ρ of $\mathbf{R}^3$, carrying p_2 into p_1, such that $\pi_u K$ surrounds $\rho\pi_u K$ at p_1. $K \cap \pi_u^{-1}[p_i]$ is, by our choice of u, a linesegment S_i of length α. By the above construction, each $q \in \text{relint}\, S_1$ is a smooth point of K; let us denote by v the outer normal of K at every $q \in \text{relint}\, S_1$. We choose an endpoint q_2 of S_2 and a point $q_1 \in \text{relint}\, S_1$. Denote by τ the translation carrying x into $x + \langle q_1 - q_2, u\rangle u$. Then there is a neighbourhood U of q_1 such that $(U \cap \tau\rho K) \subset (U \cap K)$, and $(U \cap \tau\rho K) \cap \text{bd}(U \cap K) \subset S_1$. Denote by L the line $u^\perp \cap \text{Im}(\pi_v)$. We find a rotation σ with axis $L + q_1$ and some small angle, together with a small translation ϕ in direction $-v$, such that K surrounds the convex body $\phi\sigma\tau\rho K$ at some point $q \in \text{relint}\, S_1$, close to q. By Lemma 1, K would not have the intersection property. Thus (14) is established. Now let w be a unit vector in $\mathbf{R}^3$ such that $\langle u, w\rangle = 0$. Let $X \subset \text{Im}(\pi_w)$ be a Euclidean disc congruent to $\pi_u K$, and set $Y = S_1 + X$, where S_1 is the linesegment mentioned above. Then

(15) $\pi_w K$ is parallel to Y.

Otherwise we find, by an argument similar to the one used for the proof of (14), some point $r \in \text{relbd}\, \pi_w K$ and some translate X_1 of X such that X_1 surrounds $\pi_w K$ at r. By the statement (12) of Proposition 2 there is a unique point $s \in \text{bd}\, K$ such that $\pi_w(s) = r$. Let ρ be an isometry of $\mathbf{R}^3$ such that $\rho(t) = s$ and $\pi_w[\rho K] = X_1$, where T is some linesegment in bd(K) and $t \in \text{relint}(T)$. ρK obviously surrounds K at s, and, again by Lemma 1, K would not have the intersection property. With this contradiction the statement (15) is established for every w orthogonal to u. With the aid of [8] we derive from (14) and (15) that K is homothetic to $T + B$, where T is any linesegment in bd K and B a Euclidean ball whose diameter coincides with that of $\pi_u(K)$. Proposition 3 follows.

Proposition 1 and Proposition 3 together yield our main result: If a convex body $K \subset \mathbf{R}^3$ has the intersection property, then it is either a Euclidean ball or a parallel body of some linesegment.

4. Questions

As the referee has pointed out, the following question remains open: is there a convex surface S in $\mathbf{R}^3$ which intersects each of its *directly* congruent copies in a connected set, but does not bound a parallel set of some linesegment? I think that such a surface does not exist, but have no full proof. Notice, however, that S would have to be strictly convex, since no reflections are used in the proof of Proposition 3. Under appropriate smoothness assumptions the nonexistence of S can also be established by classical differential geometric methods. I have not explored the corresponding problem in higher dimensions.

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